Abstract. The Uncertain Grid Method for PDE solution is presented. The method is a natural generalization of meshless methods based on the ideas of the free points and smooth particles hydrodynamics methods. The Uncertain Grid Method is illustrated with numerical examples of 2D gas dynamics equations and Poisson equation solution.

Key words. PDE Numerical methods, Gas Dynamics, Poisson Equation, Meshless methods

AMS subject classifications. 76M25, 65M99, 65N99

1. Introduction. There are several different approaches to the construction of numerical methods for PDE solution. The most popular approaches are grid-based methods, notably finite difference (FD) and finite element (FE), and meshless methods, notably particle-in-cell (PIC) and free-Lagrange (FL). While each method has its own advantages and disadvantages, none of them is able to accurately model some of the most difficult problems encountered in advanced scientific research.

This paper is devoted to the introduction of a new approach for the solution of PDEs, the Uncertain Grid Method (UGM). UGM builds on the advantages of meshless methods (PIC and FL) allowing the modeling of large deformations, but overcomes the disadvantages of these methods related to the nonmonotonicity and loss of accuracy of the derived numerical solutions. To help frame a discussion of the capabilities and benefits of UGM, let us begin with an overview of the traditional approaches to the solution of PDEs.

Finite difference and finite element methods are used both for stationary and moving Euler grids. There are a lot of very accurate and robust techniques based on FE and FD representation of the numerical solution but they all share certain disadvantages:

- Modeling of flows with large deformations (e.g. gas dynamics, magnetic hydrodynamics, elastic-plastic deformations) leads to a grid crash and requires the application of special algorithms for grid reconstruction. The use of these algorithms leads to a smoothing of the numerical solution and to difficulties with the description of interfaces especially in 3D cases.
- Modeling of complicated interfaces, which are common in the modeling of different instabilities, is quite difficult with FD and FE methods. The usual solution to this shortcoming lies in accounting for the concentrations of the mixing components, see [1], [2]. These methods are very complicated and have their own drawbacks. First, it is not readily evident how to modify the more or less complicated equations of state in a cell with a mixture. Second, the interface movement is also a challenge although some techniques for tracking the interface do exist.

In attempts to overcome these drawbacks the so-called meshless methods have been developed. One of the most widely used families of meshless techniques is particle-in-cell (PIC), a method initially proposed by Harlow [3]. Today there are...
numerous modifications of the PIC method, see [4], [5], [6]. A good review of PIC methods is given by Belotserkovsky and Davydov [7].

The main idea of these methods is the introduction of two grids: the Euler grid (usually rectangular) used for the evaluation of gradients, and the Lagrange grid of particles for tracking the properties of matter. These grids exchange data with each other: the matter properties are interpolated from the particles to the Euler grid, and the gradients are interpolated from the Euler grid to the particles.

The free points method (FPM) [6] is slightly different in that it only needs knowledge of the data in the neighboring points. The new values are calculated using integral presentation of the solution via values in the actual point vicinity.

A similar idea gave start to the development of a new class of PIC methods called smooth particles hydrodynamics (SPH) [8]. The main idea of these methods is based on the concept that any particle is supposed to be spread around its center with some kernel (for example, $\frac{1}{n^{2/3}h^n} \cdot e^{-\frac{(r-r_0)^2}{h^2}}$, where $n$ is the dimension of the problem). In this case the differentiating of the corresponding particle value is reduced to the differentiating of the kernel. The details of these methods can be found in [9], [10].

The main problems that appear while using these PIC methods are linked with two serious drawbacks:

- The first drawback is the necessity of approximation of the integral for FPM and approximation of the smooth function with radial basis functions. This approximation is provided with a given accuracy for an equidistant distribution of the particles, but this type of distribution can’t be achieved in the modeling of real flows with large deformations.

- The second drawback is linked with the necessity of partial derivatives approximation via approximation of the original functions. In other words, in providing a good approximation of the function in norm $C$ or $L_2$ we can’t be sure that the approximation of the derivatives is good.

There have been attempts at overcoming these drawbacks, mostly linked with different variations of SPH methods [10], but they did not result in sufficiently robust methods.

Another class of meshless methods called “Free Lagrangian” [11], [12] is based on the representation of the region by a set of Voronoi cells. Though this approach may provide robust code for modeling large deformations [13], there are two main drawbacks:

- There is no universal solution to the problem of grid construction for an arbitrary arcwise connected region $G$. While some approaches exist, they do not work in all circumstances.

- There is a problem maintaining consistency between the ”geometrical” volume and the volume obtained from the corresponding equations for volume change.

This second drawback should be explained in more detail. The equation for Lagrange volume change, $\frac{\partial V}{\partial t} = V \cdot \nabla \cdot \vec{u}$, determines the change of the specific volume depending on the velocity field. The geometrical volume in FL methods is determined from Voronoi dissection and differs from the ”gas dynamic” volume ($V = M \cdot V$ where $M$ is a mass assigned to a specific Voronoi cell) being a consequence of the Lagrange point movement described by the equation $\frac{d\vec{r}}{dt} = \vec{u}$. As can be seen, the first equation is a differential consequence of the second. The problem is that this consequence does not remain in differences. This inconsistency generates nonmonotonicity and may lead to spurious extrema.
This paper will document the construction of a meshless method that avoids the drawbacks listed above. To begin with, let us consider gas dynamics equations in Lagrange form:

\[
\frac{\partial}{\partial t} \int_V dV = \oint_{\partial V} u_n dS \\
\frac{\partial}{\partial t} \int_V \rho \cdot \vec{u} dV = -\oint_{\partial V} p \cdot \vec{n} dS \\
\frac{\partial}{\partial t} \int_V \rho \cdot (\varepsilon + \frac{1}{2} \cdot |u|^2) dV = -\oint_{\partial V} p \cdot u_n dS
\]

(1.1)

Let us suppose that we have an arbitrary distribution of points having finite volumes and all other values needed for the solution of an applied problem. These points are distributed in a finite volume bounded by a given piecewise linear surface. We suppose that the volume is convex.

To approximate 1.1 we need to get the approximation of the integral over the volume of each particle and the approximation of the boundary integrals in the right-hand sides. We seek the approximation in the form:

\[
\int_V f dV \approx \sum_{j \in \Omega_i \cup \{i\}} A_{i,j} \cdot f_j \\
\oint_{\partial V} f dS \approx \sum_{j \in \Omega_i} B_{i,j} \cdot f_{i,j}
\]

(1.2)

Here \(\Omega_i\) is a set of numbers of neighboring particles for the particle number \(i\), and \(f_{i,j}\) is the approximation of function \(f\) at the "uncertain face" between the \(i\)-th and \(j\)-th particles.

The coefficients in 1.2 should satisfy specific relationships. So for each particle we have something like a cloud of particles around it which is used to construct the approximation. As we see below, the power of this cloud (equal to the number of particles in it) can vary without significantly decreasing the quality of the approximation.

Considering the first equation in 1.2 one can easily find a good and natural approximation, \(A_{i,j} = |V_i| \delta_{ij}\), where \(\delta_{ij}\) is the Kronecker tensor. This approximation has a second order that gives a uniform distribution of the particles and is the simplest one. Higher order approximations may be useful when high order difference schemes are under construction.

In describing this new method the term "particle" is a useful description, but in reality what we have is a volume which does not have strict bounds and about which we know certain values that are necessary for modeling a specific applied problem. The unbounded nature of this volume is the basis for the name "Uncertain Grid Method".

In the remainder of this paper, Section 2 discusses the conditions on \(B_{i,j}\) for the approximation of 1.1. Section 3 examines how to get the coefficients while satisfying the conditions from Section 2. Section 4 is devoted to a discussion of the approximation of gas dynamics equations and Section 5 to the approximation of Poisson equations.

2. The Approximation Relationships. The approximation of the partial differential equations is based on the approximation of a set of basic operators. The approximation of the following operators should be considered:
\[
\begin{align*}
    f(\vec{r}_k) & \approx \frac{1}{|V_k|} \int_{V_k} f d\vec{r} \\
    \vec{\nabla} f(x_k) & \approx \frac{1}{|V_k|} \left\{ \frac{\partial}{\partial V_k} \int_{V_k} f \cdot \vec{n} dS \right\} \\
    \vec{\nabla} \cdot \vec{g}(x_k) & \approx \frac{1}{|V_k|} \left\{ \frac{\partial}{\partial V_k} \int_{V_k} \vec{g} \cdot \vec{n} dS \right\}
\end{align*}
\]

(2.1)

The approximation of the first operator in 2.1 has been discussed above. In this paper the simplest approximation of the integral operator for the \(k\)-th particle volume, \(\frac{1}{|V_k|} \int_{V_k} f dV \approx f_k\), is used.

The approximation of the divergence operator (the third operator in 2.1) is an evident consequence of the approximation of the gradient operator (the second operator in 2.1), so we will only be discussing methods of approximation of the gradient operator.

We try to find the weights \(W_{ij}\) such that the integral in the second equation in 2.1 may be approximated in the form:

\[
\left(\vec{\nabla} \cdot \vec{g}\right)(x_k) \approx \sum_{j \in \Omega_k} W_{ij} \left\| \vec{r}_j - \vec{r}_i \right\| \vec{n}_{ij} \cdot \left(\vec{\nabla} f_i + \frac{1}{2} \vec{f} \right) \]

(2.2)

The goal of our treatment in the current section is to find the relationships for \(W_{ij}\) which lead to the approximations in 2.2 with the given order. The standard procedure for the approximation treatment is used. Substitute in 2.2 \(f_i\) and \(f_j\) by the values of the approximated differentiable function \(f\) in the points \(\vec{r}_i\) and \(\vec{r}_j\) and decompose \(f_j = f(\vec{r}_j)\) in the Taylor series in the vicinity of \(\vec{r}_i\).

\[
\begin{align*}
    f_j &= f_i + \vec{\nabla} f_i \cdot (\vec{r}_j - \vec{r}_i) + \frac{1}{2} (\vec{r}_j - \vec{r}_i)^T \cdot \nabla^2 f_i \cdot (\vec{r}_j - \vec{r}_i) + O(|\vec{r}_j - \vec{r}_i|^3)
\end{align*}
\]

(2.3)

Substituting 2.3 in 2.2 and setting \(n_{ij} = \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|}\), one can get:

\[
|V_i| \cdot \langle \vec{\nabla} f \rangle_i \approx \sum_{j \in \Omega_k} W_{ij} f_i \vec{n}_{ij} + \frac{1}{2} \sum_{j \in \Omega_k} W_{ij} (\vec{r}_j - \vec{r}_i) \vec{n}_{ij} \cdot \nabla f_i + \frac{1}{2} \sum_{j \in \Omega_k} W_{ij} |\vec{r}_j - \vec{r}_i|^2 \vec{n}_{ij} \cdot \nabla^2 f_i + V_i \cdot O(d^2)
\]

(2.4)

Here \(\langle \vec{\nabla} f \rangle_i\) is the approximation of the corresponding gradient in the \(i\)-th particle, and \(d_i = \frac{\sum_{j \in \Omega_k} W_{ij} |\vec{r}_j - \vec{r}_i|^2}{\sum_{j \in \Omega_k} W_{ij} |\vec{r}_j - \vec{r}_i|}\) is the diameter of the corresponding particle. The following relationships should be fulfilled to get the first order approximation:

\[
\begin{align*}
    \sum_{j \in \Omega_k} W_{ij} \vec{n}_{ij} &= 0 \\
    \frac{1}{2} \sum_{j \in \Omega_k} W_{ij} |\vec{r}_j - \vec{r}_i| \vec{n}_{ij} \cdot \vec{n}_{ij}^T &= |V_i| \cdot I
\end{align*}
\]

(2.5)

Here \(I\) is the identity matrix.
Returning to the problem statement, we recall that initially the domain has been approximated by a set of particles. Each particle has coordinates and various physical values (depending on an applied problem), but no volumes are initially assigned to the particles. When using FE or FL methods, one can determine the corresponding volumes from geometrical considerations. The situation for UGM is different. We do not have a specific piece of space linked with a specific particle but we do have approximation of surface integrals. Remembering the formula for the volume of a closed region with measurable boundary:

\[|V| = \frac{1}{K} \oint_{\partial V} \vec{r} \cdot \vec{n} dS\]  

where \(K\) is a dimension of the space of unknowns, one can write an approximation for \(|V_i|\) in an internal particle:

\[|V_i| \approx \frac{1}{K} \sum_{j \in \Omega_i} W_{ij} \vec{n}_{ij}^T \cdot \vec{r}_{ij}\]

Here \(\vec{r}_{ij} = \frac{1}{2}(\vec{r}_j + \vec{r}_i)\). Using the first relationship in 2.5, 2.7 becomes:

\[|V_i| \approx \frac{1}{2K} \sum_{j \in \Omega_i} W_{ij} \vec{n}_{ij}^T \cdot \left(\vec{r}_j - \vec{r}_i\right)\]

Substituting 2.8 in 2.5, one can easily get the system of \(K N + K^2 N\) uniform equations, determining coefficients \(W_{ij}\):

\[\sum_{j \in \Omega_i} W_{ij} \vec{n}_{ij} = 0\]

For a 2D case, two of the six equations in 2.9 are dependent on the others. The independent set of equations is:

\[\sum_{j \in \Omega_i} W_{ij} |\vec{r}_j - \vec{r}_i| \left(n_{ij}^x n_{ij}^x - n_{ij}^y n_{ij}^y\right) = 0\]

\[\frac{1}{2} \sum_{j \in \Omega_i} W_{ij} |\vec{r}_j - \vec{r}_i| n_{ij}^x n_{ij}^y = 0\]

It is evident that the equations in 2.9 have a trivial zero solution, so we need additional equations that give us a non-trivial, non-negative solution \(W_{ij}\). To get them it is necessary to consider approximation of the boundary conditions.

Before discussion of the boundary condition approximation, let us discuss a very important issue linked with the calculation of the weights \(W_{ij}\). We have four equations for \(W_{ij}\) for each particle, so to get a solution of these equations we evidently need at least four particles in the neighborhood of the given particle. This means that at
least four \( W_{ij} \) should be nonzero for the given \( i \). Of course a larger number of non-zero weights gives us more flexibility in finding appropriate \( W_{ij} \), though it results in larger particle diameter which leads to worse approximation and to a larger number of computational operations.

To obtain an optimal solution we follow the process proposed in SPH methods, namely, the introduction of a special area of vicinity for each particle. In this case only particles from this area are considered to be the neighbors for a given particle. We consider a particle to be a neighbor for the given one if the distance between them is smaller than \( m \) average diameters:

\[
|\vec{r}_j - \vec{r}_i| \leq m \cdot (d_j + d_i)
\]

For our example we consider \( 2 \leq m \leq 4 \).

### 2.1. Boundary Conditions Approximation

As discussed earlier, the region in our example is convex and its boundary is a piecewise linear surface. Let us consider a particle that is close to the boundary. The surface integral approximation for this particle can be written as a sum of two terms:

\[
\iint_{\partial V_i} f \vec{n} dS \approx \sum_{j \in \Omega_i} W_{ij} \frac{1}{2} (f_j + f_i) \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|} + \sum_{j \in \Theta_i} W_{ij} f_j \vec{n}_j
\]

Here \( \Theta_i \) is a set of boundary elements which are located in the vicinity of the \( i \)-th particle, and \( \vec{n}_j \) is an external unit normal vector to the \( j \)-th boundary element. Each boundary element contributes value \( \sum_{j \in \Xi_i} W_{ji} \cdot f_i^B \) in the total surface integral along the external boundary. Here \( \Xi_i \) is a set of numbers of particles located in the vicinity of \( i \)-th boundary element, and \( f_i^B \) is the value of the corresponding function in the center of \( i \)-th boundary element. Based on the above formula, one should impose the corresponding restriction on \( W_{ij} \):

\[
\sum_{j \in \Xi_i} W_{ji} = S_i
\]

Here \( S_i \) is an area of the \( i \)-th boundary element.

It is necessary to define the meaning of being "located in the vicinity of" as used above. We consider that the \( i \)-th particle is in the vicinity of boundary element \( j \) if the distance from the particle to the element is less than \( m \) particle diameters, \( \vec{n}_j \leq m d_i \), and the nearest point at the boundary segment, \( \vec{r}_i^{Bj} = \vec{r}_i + (\vec{n}_j \cdot (\vec{r}_j - \vec{r}_i)) \), belongs to it.

Returning to the discussion of approximation relationships, let us discuss Equation 2.11. Substituting 2.3 in 2.11 and assigning \( f_j \) to the nearest point of the boundary segment, \( \vec{r}_i^{Bj} \), one can get the following relationships for the weights of the "boundary" particles:

\[
\sum_{j \in \Xi_i} W_{ij} \vec{n}_i \cdot (\vec{r}^T_{ij} \cdot (\vec{r}_j - \vec{r}_i)) \cdot \vec{n}_j \cdot \vec{n}^T_{ji} = |V_i| \cdot I
\]

Taking into account the equation for the volumes of the boundary particles:
particles indices

Taking the sum of the left-hand side of the second equation of the system over internal

elements 1.1 and 2.2, therefore we can leave only the equations for tensor elements
derived in the same manner.

\[ (2.15) \]

One more equation for \( W \) appears from the analysis of the system in 2.13.

Although we examine the 2D case below, the formulas for the 3D case can be
derived in the same manner.

The system of equations in 2.9 has only four independent equations because of the
symmetry of tensors in the second equation and equivalency of the equations for tensor
elements 1.1 and 2.2, therefore we can leave only the equations for tensor elements
1.1 and 1.2. Let us reformulate the system in 2.15 to get the first four equations close
to the four equations of 2.9. The resulting system is:

\[ (2.16) \]

One more equation for \( W_{ij} \) appears from the analysis of the system in 2.13.

Taking the sum of the left-hand side of the second equation of the system over internal
particles indices \( i \) one gets:

\[ (2.17) \]
The sum over the right-hand sides gives:

\( \sum_i |V_i| = |V_G| \)  \hspace{1cm} (2.18)

Here \( |V_G| \) is the volume of the region \( G \).

Let us return now to the description of the region \( G \) where the approximation takes place. We suppose that it has a piece-wise linear boundary. Each segment has the length \( S_j \), its center \( \vec{r}_j \) and the unit external normal \( \vec{n}_j \). Therefore, it is easy to see that the following formulas arise:

\[
\begin{align*}
\sum_{j \in B} x_j \cdot n^x_j \cdot S_j &= \oint_{\partial G} \vec{n} \cdot \vec{r} \, dS = 0 \\
\sum_{j \in B} y_j \cdot n^y_j \cdot S_j &= \oint_{\partial G} \vec{n} \cdot \vec{y} \, dS = 0 \\
\sum_{j \in B} S_j \cdot \vec{n}_j &= \oint_{\partial G} \vec{r} \cdot \vec{n} \, dS = 0 \\
\sum_{j \in B} x_j \cdot n^x_j \cdot S_j &= \oint_{\partial G} \vec{r} \cdot \vec{n} \, dS = 0 \\
\sum_{j \in B} y_j \cdot n^y_j \cdot S_j &= \oint_{\partial G} \vec{y} \cdot \vec{n} \, dS = 0
\end{align*}
\]  \hspace{1cm} (2.19)

System 2.19 implies:

\[
|V_G| \cdot I = \sum_{j \in B} S_j \vec{r}_j \cdot \vec{n}_j^T
\]  \hspace{1cm} (2.20)

Combining 2.17, 2.18 and 2.20, we get one more set of equations for boundary elements:

\[
S_j \cdot \vec{r}_j = \sum_{i \in \Xi_j} W_{ij} \vec{r}_i^B \hspace{1cm} \forall j \in B
\]  \hspace{1cm} (2.21)

The resulting system for the weights, determining the first order approximation for the gradients, is:

\[
\begin{align*}
\sum_{j \in \Omega_i} W_{ij} \vec{n}_{ij} + \sum_{j \in \Theta_i} W_{ij} \vec{n}_{ij} &= 0 \\
\frac{1}{4} \sum_{j \in \Omega_i} W_{ij} (\vec{r}_j - \vec{r}_i) \cdot \left( \begin{array}{c} n^x_{ij} \cdot n^y_{ij} - n^y_{ij} \cdot n^x_{ij} \\ n^y_{ij} \cdot n^y_{ij} - n^x_{ij} \cdot n^x_{ij} \end{array} \right) &= 0 \\
\frac{1}{4} \sum_{j \in \Theta_i} W_{ij} (\vec{r}_j - \vec{r}_i) \cdot \left( \begin{array}{c} n^x_{ij} n^y_{ij} - n^y_{ij} n^x_{ij} \\ n^y_{ij} n^y_{ij} - n^x_{ij} n^x_{ij} \end{array} \right) &= 0 \\
\sum_{j \in \Omega_i} W_{ij} &= S_j \\
\sum_{i \in \Xi_j} W_{ij} \vec{r}_i^B &= S_j \cdot \vec{r}_j
\end{align*}
\]  \hspace{1cm} (2.22)

In addition to approximation relationships there is one more important property of good approximators of differential operators - conservatism.
2.2. Conservatism. Conservatism is equivalent to Stoke’s theorem for the region. That means that the difference integral of the function gradient over the whole region should be equal to the difference integral of the function over the region boundary, and the expression

\[
\left( \sum_{i=1}^{N} V_i \right) \cdot \left( \sum_{j} \frac{1}{2} (f_j + f_i) \right) = \sum_{j} \frac{1}{2} (f_j - f_i) \cdot \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|} + \sum_{j \in \Theta} W_{ij} f_j \vec{n}_j
\]

should depend only on the boundary values of function \( f \).

After some simple transformations, 2.23 can be written in the form:

\[
\sum_{i=1}^{N} |V_i| \cdot \left( \sum_{(i,j) \in \mathcal{R}} (W_{ij} - W_{ji}) \frac{1}{2} (f_j + f_i) \right) \cdot \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|} + \sum_{j \in \mathcal{B}} S_j f_j \vec{n}_j = 0
\]

Here \( \mathcal{R} \) is a set of pairs of neighbor particle numbers in which \( i < j \), and \( \mathcal{B} \) is a set of numbers of boundary elements. Therefore, the symmetry of \( W_{ij} \) is enough for conservatism. That is why we consider \( W \) to be a symmetrical matrix. The main conclusion of this section can be formulated as the following theorem:

Theorem 1. Formulas 2.1 and 2.11 approximate the gradient with first order of magnitude with respect to \( \max d_i \) if the weight \( W_{ij} \) in the formulas is symmetric and satisfies the equations in 2.22.

3. The Linear System for the Weights. The goal of this section is to treat the system of equations for the weights and to propose a method for its solution. To begin with we try to treat the properties of the system of equations in 2.22. Initially it is necessary to find the rank of the system, and to educe a subsystem of the maximum rank.

3.1. The Linear System Rank. To better solve the system of equations, one needs to exclude the equations which are a consequence of other equations of the linear system. To understand whether System 2.22 has such equations or not, we sum up each left-hand side of the equations. The first pair of equations gives the following:

\[
\sum_{j \in \Omega} \left( \sum_{i,j} W_{ij} \vec{n}_i + \sum_{j \in \Theta} W_{ij} \vec{n}_j \right) = \sum_{(i,j) \in \mathcal{R}} (W_{ij} - W_{ji}) \vec{n}_j + \sum_{j \in \mathcal{B}} S_j \vec{n}_j = \sum_{j \in \mathcal{B}} S_j \vec{n}_j = 0
\]

The sum of the third and fourth equations gives us an identity as can be seen from the previous section. In additiona, we see that all four sets of equations in 2.22 are dependent, so one equation in each set should be excluded to have a non-degenerated system. For example, the equations for particle number one may be excluded.

3.2. The Statement of Problem for Weights Determination. We now have a system of equations in 2.22 for \( i = 2, ..., N \) and two conditions for \( W_{ij} \):

\[
\begin{align*}
W_{ij} &= W_{ji} \\
W_{ij} &\geq 0
\end{align*}
\]
Before calculation of the weights we must determine the weights graph. To do this we set all $W_{ij} = 0$ when the $i$-th and $j$-th particles are not neighbors or the $i$-th particle is not in the vicinity of the $j$-th boundary element.

Since $W_{ij}$ is a sparse matrix, we can enumerate the non-zero elements of the matrix $W_{ij}$ assigning the same number to the $(ij)$-th and $(ji)$-th elements using the following algorithm:

\begin{align}
J Nu & = 1; \\
& \text{for } i = 1 \text{ by } 1 \text{ to } N \\
& \quad \text{for } j = i + 1 \text{ by } 1 \text{ to } N \\
& \quad \quad \text{if } (W_{ij} \neq 0) \\
& \quad \quad \quad \text{then } Numb_{ij} = J Nu; \\
& \quad \quad \quad J Nu = J Nu + 1; \\
& \quad \quad \text{endif} \\
& \text{end for} \\
& \text{end for} \\
\end{align} \tag{3.3}

Enumeration algorithm 3.3 is implemented more effectively by application of its inner cycle only to non-zero elements of $W$. As a result of 3.3, the sparse integer matrix $Numb_{ij}$ is used for 1D enumeration and the linear system 2.22 may be rewritten with respect to one-dimensional vector $\alpha$ in the form:

\begin{align}
A \cdot \alpha = b \\
\end{align} \tag{3.4}

Here $\alpha_{Numb_{ij}} = W_{ij} = W_{ji}$. This formulation gives us the symmetrical solution $W_{ij}$ of the original linear problem 2.22.

There is one more restriction to the system 3.4:

\begin{align}
\alpha \geq 0 \\
\end{align} \tag{3.5}

As we can see, the solution of 3.4 with restrictions 3.5 looks like a linear programming (LP) problem for minimizing $\sum_i c_i \cdot \alpha_i$ under restrictions 3.4 and 3.5 where $c_i$ are positive coefficients. The only difference is that we do not need to get the "exact" minimum. The approximation takes place for any "interior" point, that is any point satisfying restrictions 3.4 and 3.5. The constant in approximation estimation will depend on how close we are to the exact minimum. From a practical point of view we do not need to come very close to the solution of the LP problem. The choice of coefficients $c_i$ has been treated in [17]. It was shown that the best choice is:

\begin{align}
c_{Numb_{ij}} = |\vec{r}_i - \vec{r}_j|^2 \\
\end{align} \tag{3.6}

The theory of LP can be found for example in [15], [16]. Below, the algorithm used by the author is described. We use here the primal-dual logarithmic barrier method which results in the following linear system:

\begin{align}
A^T \cdot y + s &= c \\
A \cdot \alpha &= b \\
D_s \cdot D_\alpha \cdot e &= \mu \cdot e \\
\end{align} \tag{3.7}
Here \( D_s = \text{Diag}(s), D_{\alpha} = \text{Diag}(\alpha) \) and \( e = (1, 1, ..., 1)^T \). The solution of 3.7 streams to the optimal one when \( \mu \to 0 \). As noted above, we do not need a "very" optimal solution, so we can assume \( \mu \) is equal to a constant, i.e. 1, and solve the corresponding nonlinear equations in 3.7.

To solve 3.7 it is reasonable to use Newton method for a consequence of 3.7:

\[
A^T \cdot y + \mu \cdot D^{\omega - 1} \cdot e = c
\]
\[
A \cdot \alpha = b
\]

Let us suppose that the \( \nu \)-th iteration is done, so we have \( y^\nu \) and \( \alpha^\nu \), now the values at the \((\nu + 1)\) iteration can be derived from the system:

\[
A^T \cdot y^{\nu + 1} - \mu \cdot D^{\omega - 2} \cdot (\alpha^{\nu + 1} - \alpha^\nu) = (c - \mu \cdot D^{\omega - 1} \cdot e)
\]
\[
A \cdot (\alpha^{\nu + 1} - \alpha^\nu) = b - A \cdot \alpha^\nu
\]

After simple transformations, system 3.9 reduces to the system for \( y \):

\[
A \cdot D^{\omega - 2} \cdot A^T \cdot y^{\nu + 1} = \mu(b - A \cdot \alpha^\nu) + A \cdot D^{\omega - 2} \cdot (c - \mu^{-1} \cdot D^{\omega - 1} \cdot e)
\]

The solution to \( y^{\nu + 1} \) of 3.10 is substituted in the first equation of 3.9 and vector \( \alpha^{\nu + 1} \) is obtained.

4. Central Difference Scheme for Gas Dynamics. This section describes the Uncertain Grid Central Difference scheme (UGCD) for gas dynamic equations. The scheme construction is based on ideas from the paper [14]. In this paper we describe only the first order variant of UGCD. The results are illustrated with 1D and 2D numerical examples.

The equations of Lagrangian gas dynamics can be written in the form:

\[
\frac{\partial}{\partial t} \int_V dV = \oint_V \frac{\partial}{\partial \nu} u_n dS
\]
\[
\frac{\partial}{\partial t} \int_V \rho \cdot \bar{u} dV = - \oint_V \frac{\partial}{\partial \nu} p \cdot \bar{n} dS
\]
\[
\frac{\partial}{\partial t} \int_V \rho \cdot (e + \frac{1}{2} \cdot |\bar{u}|^2) dV = - \oint_V \frac{\partial}{\partial \nu} p \cdot u_n dS
\]

Let us suppose that the region is bounded either by a rigid wall (\( \bar{u} \cdot \bar{n} = 0 \)) or piston (given (\( \bar{u}^T \cdot \bar{n} \))), We use the first order formulation of the KT scheme. The scheme can be written in the form:
The number of particles in the x-direction was initially equal to 20, each 1D interval is
boundaries the condition is a "rigid wall" that is given the velocity $\vec{u}_0$. Initially, the gas in region $0 \leq y \leq 2.5$ has the properties $\rho = 4.0, \varepsilon = 0.5, u_y = -1.0$ and $\gamma = 5/3$. Gas in the region $0 \leq y \leq 1.5$ has the properties $\rho = 4.0, \varepsilon = 0.5, u_y = -1.0$ and $\gamma = 5/3$. The results are presented in Figures 4.1 - 4.4.
represented by 20 points in the figures. As we can see, there is some nonmonotonicity at the shock wave while the number of 1D intervals at the shock is about 3. To avoid such nonmonotonicity the author proposes to use entropy viscosity.

4.1. Entropy Viscosity. The equations in 4.1 are the equations of isentropic flow, so the equation for the entropy should be fulfilled as:

\[ T \cdot dS = dE + P \cdot dV \]

Here \( V \) is a specific volume: \( V = \frac{1}{\rho} \). At the shock wave the entropy takes a jump. This increase in entropy is the main reason for the nonmonotonicity. Based on this it is reasonable to introduce the viscosity which smoothes the entropy discontinuity. The central scheme of the equation for entropy with such viscosity looks like:

\[ T^n_i \cdot (S_i^{n+1} - S_i^n) = \frac{\tau}{2} \cdot \sum_{j \in \Omega_i} W_{ij} \cdot c^n_{ij} \cdot \left( \frac{T^n_j S^n_j}{V_j} - \frac{T^n_i S^n_i}{V_i} \right) \]  

As we do not have an entropy variable, it is reasonable to distribute the entropy viscosity change between energy and volume (density). This can be done in the following way. Let us add the equation for the pressure increment \( dP \):

\[ dP = P_z \cdot dE + P_V \cdot dV \]

to 4.3. Having \( dP \) and \( T \cdot dS \) given, we get the following expressions for \( dV \) and \( dE \):

\[ dE = \frac{T \cdot dS - dP/(\rho P_V)}{1-P_z/P_V} \]
\[ dV = \frac{dP/P_V - P_z/P_V \cdot T \cdot dS}{1-P_z/P_V} \]

The formulas in 4.6 give us an idea of entropy viscosity distribution:

\[ E_{ent} = \frac{\tau}{2} \cdot \sum_{j \in \Omega_i} W_{ij} \cdot c^n_{ij} \cdot \left( \frac{T \cdot dS - dP/(\rho P_V)}{1-P_z/P_V} \right)_{ij} \]
\[ V_{ent} = \frac{\tau}{2} \cdot \sum_{j \in \Omega_i} W_{ij} \cdot c^n_{ij} \cdot \left( \frac{dP/P_V - P_z/P_V \cdot T \cdot dS}{1-P_z/P_V} \right)_{ij} \]
Here \( \{T dS\}_{ij} = (E_j - E_j) - P_{ij} \cdot (V_j - V_i) \) and \( dP_{ij} = P_j - P_i \). The upper indices are omitted as all values are calculated at the same n-th time step. The results of modeling the first problem are presented in Figures 4.7 - 4.8.

As we can see in these figures, the entropy viscosity application leads to a smoother solution conserving a rather sharp wavefront.

### 4.2. Correlation of Volume and Coordinates Change

One more issue that appears while using the described method is linked to the noncorrelation of the two equations for coordinates and specific volumes. The equation for specific volume is a consequence of the equation for coordinates as the specific volume is determined by:

\[
\frac{V(\xi,t)}{V(\xi,0)} = \nabla \xi \cdot \vec{r}
\]  

(4.8)

Here \( \vec{\xi} \) is the vector of Lagrange coordinates. Looking at the two equations below:

\[
\frac{\partial V}{\partial t} = V \nabla \vec{\xi} \cdot \vec{u}
\]

\[
\frac{\partial V}{\partial \xi} = \frac{\partial}{\partial \xi} \nabla \vec{\xi} \cdot \vec{u}
\]

(4.9)
we see that by taking the divergence of the second equation, we get the first equation. As such, the viscosity added to the first equation in 4.2 should correlate to the viscosity in the second equation. To obtain such correlation, the equation for coordinates at the time $t^n$ may be written in the form:

$$
\frac{\partial \tilde{u}_i^n}{\partial t} = \tilde{u}_i^n + \delta \tilde{u}_i^n
$$

The term $\delta \tilde{u}_i^n$ relates to the viscosity. Presenting $\delta \tilde{u}_i^n = \nabla \phi^n$ and substituting it in the first equation of 4.9, we get:

$$
\mathcal{V}_i^n \cdot \Delta \phi^n = \frac{\partial \phi^n}{\partial t} - \mathcal{V}_i^n \cdot \tilde{\nabla} \cdot \tilde{u}_i^n
$$

The boundary condition for 4.11 is written in the natural form:

$$
\phi^n|_G = 0
$$

Equation 4.11 with boundary condition 4.12 is solved using a difference scheme described in the next section. After obtaining the solution, $\delta \tilde{u}_i^n$ is calculated:

$$
\delta \tilde{u}_i^n = \frac{1}{|V_i^n|} \cdot \sum_{j \in \Omega_i} W_{ij} \cdot \frac{\phi_j + \phi_i}{2} \cdot \frac{\tilde{r}_j - \tilde{r}_i}{|\tilde{r}_j - \tilde{r}_i|} + \frac{1}{|V_i^n|} \cdot \sum_{j \in \Theta_i} W_{ij} \phi_i \cdot \tilde{n}_j
$$

Then $\delta \tilde{u}_i^n$ is substituted in Equation 4.10 and the new coordinates are evaluated.

**Remark 1.** Equation 4.11 with boundary condition 4.12 has a one-dimensional kernel. To get a correct solution to the problem, one may just set $\phi$ to a given constant at a given point, i.e. $\phi(\tilde{r}_1) = 1$. This means that we solve the corresponding difference equations for all particles besides the first one.

**Remark 2.** It is reasonable to ask why we need to use such a complicated procedure for coordinate viscosity calculation. One answer is that the usual practice for Lagrange methods is the elimination of the equation for specific volumes and calculation cells (or elements) volumes via coordinates. This procedure works well for the methods with fixed vicinity (fixed topology of the difference stencil).

With regard to irregular methods (FL, PIC or SPH), we should consider the reconfigurable stencil. In this case the correctness of the procedure of volume calculation using the coordinates of the cell (or element) nodes has to be proven. Some methods do not eliminate the equation for volumes and directly calculate the specific volume using a difference scheme. In this case a conflict between differential and difference restrictions may appear. That is, 4.8 is not fulfilled in difference form due to viscosity terms. To avoid this conflict the described procedure of introducing numerical viscosity in the equations for coordinates has been proposed.

### 4.3. Example Using 2D Richtmayer - Meshkov Instability Calculation.

It is well known that Richtmayer-Meshkov instability (RMI) takes place when a shock wave passes an interface between liquids (gases) having different densities. The RMI modeling in this paper is presented as a sample usage of the proposed algorithms for modeling flows with large deformations.

Initially light gas is located in the region $\cos(x) \leq y \leq 9$ and heavy gas is located in the region $-9 \leq y \leq \cos(x)$. No gravity is taken into account. The boundary
condition at the upper bound is a "piston" with normal velocity equal to -0.3. The boundary condition at the other boundaries is a "wall". The shock wave passes the interface several times leading to an increase in the initial perturbation. The initial matter distribution is shown in Figure 4.9. Subsequent distributions at various times are shown in Figures 4.10 - 4.11.

As we can see in the figures, the results are reasonable. It should be noted that there was no manual intervention in the calculations which shows the strong robustness of the method.

5. Difference Scheme for Poisson Equation. In this section the difference scheme for the Poisson equation for UGM is discussed. The use of the described approach allows for developing difference schemes for elliptic and parabolic (diffusion) equations.

We consider the Poisson equation with Dirichlet boundary conditions. The region $G$, where the solution is being sought, is bounded by a closed, piece-wise linear curve. The equation has the form:

\[
\Delta u = f \\
u|_{\partial G} = u_0
\]

(5.1)

If $u_0$ and $\partial G$ are smooth enough, function $u_0$ can be continued smoothly in $G$
and Equation 5.1 for \( u \) can be written for \( \hat{u} = u - u_0 \) where \( u_0 \) is a continuation of \( u_0 \) in \( G \):

\[
\Delta \hat{u} = \hat{f} \\
\hat{u}|_{\partial G} = 0
\]  

(5.2)

Here \( \hat{f} = f - \Delta \hat{u} \).

Below we consider Equation 5.2 and approximate this equation using the Uncertain Grid approach. We omit "hats" in variables \( \hat{u} \) and \( \hat{f} \) below. The natural approximation of the Poisson equation can be written in the form:

\[
\sum_{j \in \Omega_i} W_{ij} \frac{u_j - u_i}{|\vec{r}_j - \vec{r}_i|} + \sum_{j \in \Theta_i} W_{ij} < \nabla n u >_{i} |V_i| - |V_i| \cdot \vec{n}_j = 0
\]

(5.3)

Here \( < \nabla n u >_{i} \) is a second order approximation of the normal gradient of the unknown function \( u \) at the point \( t_i = \vec{r}_i + (\vec{r}_j - \vec{r}_i)^T \vec{n}_j \), that is: \( < \nabla n u >_{i} \approx \nabla n u(\vec{r}_i) + O(|d_j|^2) \) where \( d_j \) is a diameter of the auxiliary point \( \vec{r}_j \). For the Neumann boundary condition this approximation is natural as the normal gradient is given.

For the Dirichlet boundary condition at 5.2 the following technique is used. To get a second order approximation of the normal gradient, let us find the approximation in the following form:

\[
< \nabla n u >_{i} \approx 2 \frac{u_j - u_i}{|\vec{r}_j - \vec{r}_i|} - < \nabla u >_{i} \cdot \vec{n}_j
\]

(5.4)

The approximation of \( < \nabla u >_{i} \) we take in the same form as earlier:

\[
|V_i| < \nabla n u >_{i} \approx |V_i| \nabla u(\vec{r}_i) + \sum_{j \in \Omega_i} W_{ij} |\vec{r}_j - \vec{r}_i| \left( \vec{n}_j^T \nabla^2 u(\vec{r}_i) \cdot \vec{n}_j \right) \vec{n}_j + \sum_{j \in \Theta_i} W_{ij} u_j \vec{n}_j
\]

(5.5)

Substituting Taylor series for \( u_j \) in 5.5 and applying 2.22 we get:

\[
|V_i| < \nabla n u >_{i} \approx |V_i| \nabla u(\vec{r}_i) + \sum_{j \in \Omega_i} W_{ij} |\vec{r}_j - \vec{r}_i| \left( \vec{n}_j^T \nabla^2 u(\vec{r}_i) \cdot \vec{n}_j \right) \vec{n}_j + |V_i|O(|d_j|^2)
\]

(5.6)

To get the second order approximation it is necessary to have the second bracket in 5.6 be equal to zero. This gives us four more equations for \( W_{ij} \) in the particle located in the boundary vicinity:

\[
\sum_{j \in \Omega_i} W_{ij} |\vec{r}_j - \vec{r}_i|^2 (\vec{n}_j)_x^2 + \sum_{j \in \Theta_i} W_{ij} |\vec{r}_j - \vec{r}_i|^2 (\vec{n}_j)_x^2 = 0
\]

(5.7)

\[
\sum_{j \in \Omega_i} W_{ij} |\vec{r}_j - \vec{r}_i|^2 (\vec{n}_j)_y^2 + \sum_{j \in \Theta_i} W_{ij} |\vec{r}_j - \vec{r}_i|^2 (\vec{n}_j)_y^2 = 0
\]
Adding 5.7 to 2.22 we get the resulting system of equations for the weight of the graph of vicinity \( W_{ij} \) which gives us the second order approximation of the gradient at the boundary. Applying this approximation to 5.4 we get the second order approximation of the normal gradient at the boundary point \( \vec{r}_i^{B_j} \) thus getting a good approximation of the Dirichlet boundary condition for the Poisson equation.

Let us now treat the approximation and stability of the presented difference scheme 5.3, 5.4 and 5.5.

5.1. Approximation of the Poisson Equation. Let us consider 5.3 for an internal point. Presenting \( u(\vec{r}_j) \) for internal particles and \( \nabla_n u(\vec{r}_i^{B_j}) \) at boundary elements as Taylor series:

\[
\begin{align*}
\nabla_n u(\vec{r}_i^{B_j}) &= \nabla u(\vec{r}_i^{B_j}) \cdot (\vec{n}_j) + \frac{1}{2} (\vec{r}_j - \vec{r}_i)^T \cdot (\nabla^2 u)(\vec{r}_i^{B_j}) \cdot (\vec{r}_j - \vec{r}_i) + O(|\vec{r}_j - \vec{r}_i|^3) \\
\n\nabla_n (\vec{r}_i^{B_j}) &= \nabla u(\vec{r}_i^{B_j}) \cdot (\vec{n}_j) + \frac{1}{2} ((\vec{r}_j - \vec{r}_i)^T \cdot \vec{n}_j)^2 \cdot (\nabla^2 u)(\vec{r}_i^{B_j}) \cdot \vec{n}_j + O(|(\vec{r}_j - \vec{r}_i)^T \cdot \vec{n}_j|^2 + d_i^2)
\end{align*}
\]

and substituting 5.8 in the left-hand side of 5.3, we get:

\[
(\nabla u)^T \left( \sum_{j \in \Omega_i} W_{ij} \vec{n}_j + \sum_{j \in \partial \Omega_i} W_{ij} \vec{n}_j \right) + Tr \left( \nabla^2 u \cdot \left\{ \frac{1}{2} \sum_{j \in \Omega_i} W_{ij} (\vec{r}_j - \vec{r}_i) \vec{n}_j \vec{n}_j^T \right\} \right) - |V_i| f_i + |V_i| O(d_i)
\]

Here \( Tr(Q) \) is the trace of a tensor \( Q \) that is the sum of the elements at the main diagonal of the tensor. After simple transformations we can reduce 5.9 to the form:

\[
|V_i| Tr \left( \nabla^2 u \cdot I \right) - |V_i| f_i + |V_i| O(d_i) = |V_i| \left( \Delta u_i - f + O(d_i) \right)
\]

5.10 means that the Poisson equation is approximated by 5.3 with first order with respect to the maximum diameter of the particles. To finalize the treatment of the approximation we need to prove that 5.4 has second order of magnitude with respect to \( d_i \). Decomposing \( u(\vec{r}_i) \) and \( (\nabla u)^T (\vec{n}_i) \) in the vicinity of \( \vec{r}_i^{B_j} \) we get:

\[
\begin{align*}
2 \cdot \left( \nabla u(\vec{r}_i^{B_j})^T \cdot \vec{n}_j + \frac{1}{2} |\vec{r}_i^{B_j} - \vec{r}_i|^2 \cdot \nabla^2 u(\vec{r}_i^{B_j}) \cdot \vec{n}_j + O(|\vec{r}_i^{B_j} - \vec{r}_i|^2) \right) &
-
\nabla u(\vec{r}_i^{B_j})^T \cdot \vec{n}_j + |\vec{r}_i^{B_j} - \vec{r}_i|^2 \cdot \nabla^2 u(\vec{r}_i^{B_j}) \cdot \vec{n}_j + O(|\vec{r}_i^{B_j} - \vec{r}_i|^2)
= \nabla u(\vec{r}_i^{B_j})^T \cdot \vec{n}_j + O(|\vec{r}_i^{B_j} - \vec{r}_i|^2)
\end{align*}
\]

This proves the second theorem:

Theorem 2. The difference scheme for the Poisson equation 5.3 - 5.4 approximates the original equation with first order with respect to the maximum diameter of the particles.
5.2. Stability of the Difference Scheme. Let us suppose that the solution is found in the unit square. The difference scheme for 5.3 can be written in the form:

\[ R_i \cdot u_i - \sum_{j \in \Omega_i} Q_{ij} \cdot u_j = b_i \]

We need to treat coefficients \( R_j \) and \( Q_{ij} \) for the particles. The difference scheme for the values in the particles may be written in the following form:

\[ \sum_{j \in \Omega_i} W_{ij} \left( \frac{u_i - u_j}{|\vec{r}_j - \vec{r}_i|} \right) + \sum_{j \in \Psi_i} W_{ij} \left( 2 \frac{u_i - u_j}{|\vec{r}_j - \vec{r}_i|} - \frac{1}{|V_i|} \sum_{k \in \Theta_k} W_{ik} \frac{(\vec{n}^T_k \cdot \vec{n}_j)}{2} \right) = 0 \]

Here \( \Theta_i \) is the set of the numbers of the Dirichlet boundary elements in the vicinity of the \( i \)-th particle, and \( \Psi_i \) is the set of the numbers of the Neumann boundary elements. Let us combine the terms in 5.13 referring to the differences \( u_j - u_i \) where \( u_j \) refers to an internal particle:

\[ \sum_{k \in \Theta_i} W_{ik} \frac{u_k - u_i}{|\vec{r}_k - \vec{r}_i|} \left( 1 - \frac{1}{|V_i|} \sum_{j \in \Theta_i} W_{ij} (\vec{n}^T_k \cdot \vec{n}_j) \right) |\vec{r}_k - \vec{r}_i| \]

Considering the last sum in the brackets, we can see that taking into account the convexity \( G \):

\[ \sum_{j \in \Theta_i} W_{ij} (\vec{n}^T_k \cdot \vec{n}_j) |\vec{r}_k - \vec{r}_i| = \sum_{j \in \Theta_i} W_{ij} (\vec{r}_k - \vec{r}_i)^T \vec{n}_j \leq |V_i| \]

This means that the coefficient in 5.14 before the difference \( u_j - u_i \) is positive.

Now let us calculate the coefficient at the difference between the values in the particle and a boundary element:

\[ \sum_{j \in \Theta_i} W_{ij} \left( \frac{u_i - u_j}{|\vec{r}_j - \vec{r}_i|} \right) \left( 2 - \frac{1}{|V_i|} \sum_{k \in \Theta_k} W_{ik} (\vec{r}_j^B_k - \vec{r}_i)(\vec{n}^T_k \cdot \vec{n}_j) \right) \]

Let us consider the last sum in 5.16. After some self-evident transformations we get:

\[ \frac{1}{|V_i|} \sum_{k \in \Theta_k} W_{ik} (\vec{r}_j^B_k - \vec{r}_i)(\vec{n}^T_k \cdot \vec{n}_j) = \frac{1}{|V_i|} \sum_{k \in \Theta_k} W_{ik} (\vec{r}_j^B_k - \vec{r}_i)^T \vec{n}_k = \]

\[ \frac{1}{|V_i|} \sum_{k \in \Theta_k} W_{ik} (\vec{r}_j^B_k - \vec{r}_i)^T \vec{n}_k + \frac{1}{|V_i|} \sum_{k \in \Theta_k} W_{ik} (\vec{r}_j^B_k - \vec{r}_i^B_k)^T \vec{n}_k \]

As mentioned above in Section Two, a particle has non-zero weight with a boundary element only if \( \vec{r}_i^B_k \in G \). In this case the following holds true: \( (\vec{r}_j^B_k - \vec{r}_i^B_k)^T \vec{n}_k \leq 0 \).
This means that the sum in 5.17 is less than 1 and the coefficients before the difference \( u_j - u_i \) in sum 5.16 is positive.

The calculations above lead us to the following conclusion. The coefficients \( R_i \) and \( Q_{ij} \) are nonnegative and satisfy the equation, \( R_i \geq \sum_{j \in \Omega_i} Q_{ij} \), and there exists \( i \) for which the last inequality becomes strict. This fact implies the following theorem.

Theorem 3. Any solution of the system of equations in 5.3 is bounded: \( |u_i| \leq \max_k |f_k| \).

The proof of the theorem is obtained using the standard technique, see for example [18]. The theorem means that the numerical solution is stable in \( C \) norm. According to the Lax theorem, Theorems 2 and 3 imply the convergence of the difference scheme for the Poisson equation for the Uncertain Grid Method.

5.3. Numerical Example. To understand the real properties of the proposed method, we solved a sample problem that has an exact solution:

\[
\Delta u = -2 \cdot \pi^2 \sin \pi x \cdot \sin \pi y
\]

The solution is sought in the unit square with the uniform Dirichlet boundary conditions. The solution has the form:

\[
u = \sin \pi x \cdot \sin \pi y
\]

Initially all particles have been distributed uniformly; 40 in the \( x \)-direction by 40 in the \( y \)-direction. The numerical results for the problem are shown in comparison with the exact solution in Figures 5.1 and 5.2.

The exact solution is presented by the solid lines in the Figures, and numerical solution is presented by the squares. As can be seen from the figures, there is close agreement between the numerical results and the exact solution.

To get the numerical order of accuracy of the difference scheme, a couple of additional calculations for different numbers of particles have been performed. Three more numerical statements have been considered: 10x10 particles, 20x20 particles and
80x80 particles. The results of modeling have been compared to the numerical results using $L_2$ norm. The resulting error norms are presented in Table 5.1.

<table>
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<th>Diam</th>
<th>10x10</th>
<th>20x20</th>
<th>40x40</th>
<th>80x80</th>
</tr>
</thead>
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<td>2</td>
<td>1.962e-2</td>
<td>4.028e-3</td>
<td>9.860e-4</td>
<td>1.790e-4</td>
</tr>
<tr>
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<td>9.203e-3</td>
<td>2.021e-3</td>
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<tr>
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<td>6.827e-2</td>
<td>1.499e-2</td>
<td>3.894e-3</td>
<td>7.264e-4</td>
</tr>
</tbody>
</table>

Table 5.1. Numerical errors for Poisson equation solution

As we can see, the proposed scheme has second order accuracy for a uniform distribution of particles. Of course such distribution cannot be achieved in real modeling, so the question of how the scheme behaves for an arbitrary distribution is worth investigating. To answer this question the following approach was taken.

Starting with initial uniform particle distribution, the particles were moved chaotically. At each step of the process the Poisson equation with the same right-hand side 5.18 and the same boundary conditions 5.2 was solved and the error calculated using $C$ norm. We can see in Figure 5.3 the distribution of particles after 20 random steps. The error dependency at each iteration step is shown in Figure 5.4. As we can see, the error growth is less than two times. It is mostly linked with the growth of the maximum diameter of the particles $d_i$, thus enlarging the approximation error.

![Fig. 5.3. Distribution of particles in 20 steps](image1)

![Fig. 5.4. Error dependence on the iteration](image2)

One more observation is linked with the choice of the area of vicinity. The first column in Table 5.1 refers to the number of diameters $m$ used for the preliminary determination of the vicinity graph. As can be seen, the dependence on this parameter is not very significant, however, the larger the search area is for the neighboring particles, the larger the approximation error.

6. Conclusion. The Uncertain Grid Method for PDE approximation can be used successfully for the modeling of flows with large deformations, especially when it is necessary to take into account many physical processes involving the solution of parabolic and elliptic equations (i.e. MHD). It can be also used for the solution of PDEs in complicated regions.

The theoretical treatment and initial numerical experiments with the method described in this paper allow the author to make the conclusion that the proposed method has good approximation properties. It also may be applied to the construction
of high order difference schemes just by expanding the set of equations for the weights of the vicinity graph.

Of course all of the advantages that UGM has over the traditional FD and FE methods do not come for free - the payment is the computational cost of the solution of the corresponding LP problem. If one has a problem which can easily be solved using the traditional FD or FE methods, those methods should be used. However, when the problems of modeling large deformations or complicated interfaces appear, UGM can provide improved accuracy and robustness, and the increased computational costs can be offset by eliminating manual intervention in the process of calculations.

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